# IV Introduction to Spectral Sequences

long exact sequences are a great algebraic tool to try to compute (co)homology, e.g. L.E.S. of a pair $(X,A)$

Spectral sequences are a more involved algebraic gaget to make more involved computations

## A Algebraic Spectral Sequences

a bigraded module (complex...) is an indexed collection of modules $E_{s,t}$ for every pair of integers $s,t$

a differential of bidegree $(-r, r-1)$ is a collection of homomorphisms

$$d: E_{s,t} \longrightarrow E_{s-r,\,t+r-1}$$

for all $s,t$ such that

$$d^2 = 0$$

the homology of $d$ is the bigraded module

$$H_{s,t}(E,d) = \frac{\ker(d: E_{s,t} \longrightarrow E_{s-r,t+r-1})}{\operatorname{im}(d: E_{s+r,t-r+1} \longrightarrow E_{s,t})}$$

note: if we set $E_q = \bigoplus_{s+t=q} E_{s,t}$ then $d$ induces a homomor.

$$\partial : E_q \to E_{q-1}$$

and $(E_q, \partial)$ is a chain complex moreover, $H_q(E_*, \partial) = \bigoplus_{s+t=q} H_{s,t}(E, d)$

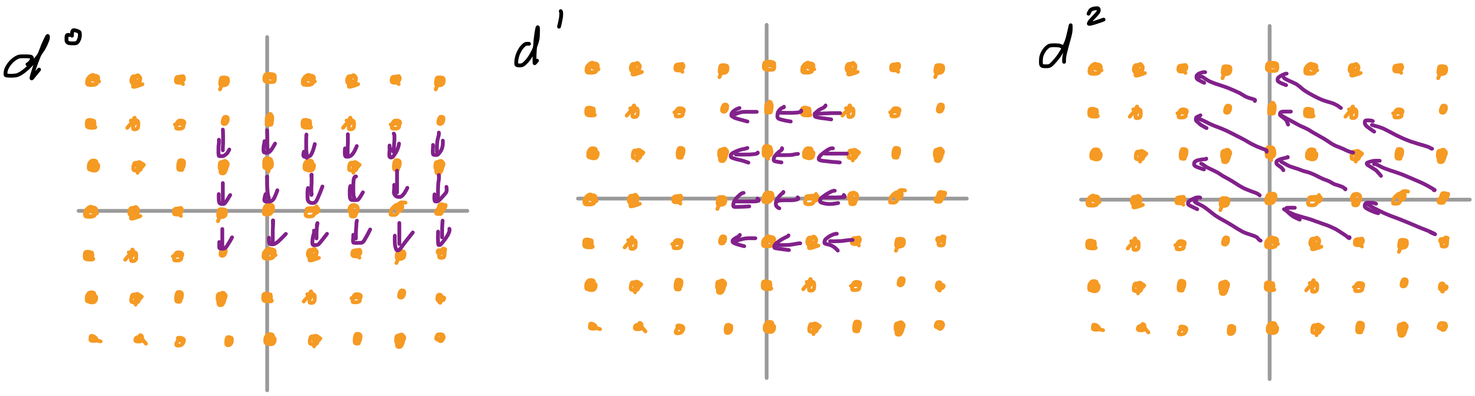
an $E^k$-spectral sequence $E$ is a seqence $\{E^r, d^r\}$

for $r \geq k$ s.t.

a) $E^r$ is a bigraded module and $d^r$ is a differential of bidegree $(-r, r-1)$

b) $E^{r+1} = H_*(E^r, d^r)$ for $r \geq k$

example:




suppose for every $s,t$ there is some $r(s,t)$ s.t.

$\forall r > r(s,t)$

$$d^r : E^r_{s,t} \to E^r_{s-r,\, t+r-1}$$

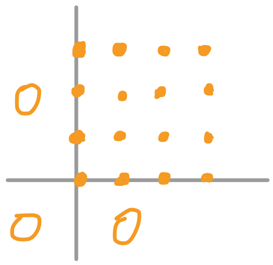
is the zero map

then $E^{r+1}_{s,t}$ is just a quotient of $E^r_{s,t}$ $\forall r \geq r(s,t)$

we can define $E^\infty_{s,t}$ = direct limit $E^r_{s,t}$

in this situation we say the spectral sequence converges to the bigraded complex $E^\infty$

note: if $E^r_{s,t} = 0$ for all $s,t<0$ (1st quadrant s.s.)




then for all $s,t$, $\exists\, r(s,t)$ s.t. $E^r_{s,t} = E^{r+1}_{s,t}$ $\forall r \geq r(s,t)$

since arrows get longer as $r$ increases!

So where do spectral sequences come from?

Filtrations!

a filtration $F$ of a module $A$ is a sequence of submodules $\{F_s A\}$ of $A$ s.t.

$$A \supset \cdots \supset F_{s+1} A \supset F_s A \supset \cdots$$

if $A$ is graded $\{A_k\}$ then the filtration inherits a grading

$$F_s A = \{F_s A_k\}$$

the associated graded (or adjoint) module is

$$G(A)_s = F_s A / F_{s-1} A$$

$$G(A) = \bigoplus_s G(A)_s$$

if $A$ is graded then $G(A)$ is bigraded

$$G(A)_{s,t} = F_s A_{s+t} / F_{s-1} A_{s+t}$$

$s$ is called the filtration degree

$t$ " " " " complementary degree

$s+t$ " " " " total degree

a filtration is convergent if $\bigcap_s F_s A = 0$ and $\bigcup_s F_s A = A$

(we will usually have a finite filtration

$$A = F_n A \supset \dots \supset F_0 A \supset F_{-1} A = 0)$$

note: even for finite fitrations $G(A)$ does not determine $A$

eg. 1) let
$$\begin{array}{c} A = A_1 = \mathbb{Z}_4 \\ \cup \\ A_0 = \mathbb{Z}_2 \\ \cup \\ A_{-1} = 0 \end{array}$$

now
$$G(A)_s = \begin{cases} \mathbb{Z}_2 & s = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

and
$$\begin{array}{c} A' = A_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \cup \\ A_0 = \mathbb{Z}_2 \\ \cup \\ A_{-1} = 0 \end{array}$$

so
$$G(A')_s = \begin{cases} \mathbb{Z}_2 & s = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

2) similarly
$$\begin{array}{c} A = A_1 = \mathbb{Z} \\ \cup \\ A_0 = 2\mathbb{Z} \\ \cup \\ A_{-1} = 0 \end{array}$$

then $G(A) = {G(A)_1}/{G(A)_0} \oplus {G(A)_0}/{G(A)_{-1}}$

$\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \ncong A$

but $G(A)$ is "close" to $A$

**lemma 1:**

if $F$ is a finite filtration of $A$, then

1) if $G(A)_s$ is free for all $s$ then $G(A) \cong A$
2) if $G(A)_s$ is a vector space over a field then
   $G(A) \cong A$
3) if $G(A)_s$ is finite for all $s$ then $A$ is finite
   and $\operatorname{order}(A) = \operatorname{order}(G(A))$
4) if $G(A)_s$ is finitely generated for all $s$, then
   $A$ is finitely generated and $\operatorname{rank}(A) = \operatorname{rank}(G(A))$
5) if $G(A)_s = 0$ for all but one $s$, then $A = G(A)$
6) if $G(A)_s = 0$ for all but two $s$, say $G(A)_k, G(A)_\ell$ with $k<\ell$, then
   $0 \to G(A)_k \to A \to G(A)_\ell \to 0$
   is exact

**Proof:**

1) we have $0 \to F_{n-1}(A) \to \underset{\substack{\cong \\ A}}{F_n(A)} \to \underset{\substack{\cong \\ G(A)_n}}{F_n A / F_{n-1} A} \to 0$

**exercise:** if $0 \to B \to A \to C \to 0$ is exact and $C$ free, then $A = B \oplus C$

so $A \cong G(A)_n \oplus F_{n-1}(A)$

similarly $F_{n-1}(A) \cong G(A)_{n-1} \oplus F_{n-2}(A)$

and so on

so $A \cong \bigoplus_s G(A)_s$

5) $A = F_n(A) \supset \dots \supset F_{-1}(A) = 0$

and $G(A)_s = \begin{cases} F_s(A) & s = k \\ 0 & s \neq k \end{cases}$

so $G(A) = G(A)_k$

and $F_n(A) = F_{n-1}(A) = \dots = F_k(A) \supset F_{k-1}(A) = \dots = F_0(A) = 0$

so $A = F_k(A) = G(A)_k = G(A)$

6) if $G(A)_s = \begin{cases} C & s = l \\ B & s = k \\ 0 & s \text{ otherwise} \end{cases}$

then $A = F_n(A) \dots = F_l(A) \supset F_{l-1}(A) = \dots F_k(A)$

$\supset F_{k-1}(A) = \dots = F_1(A) \supset F_{-1}(A) = 0$

note: if $k \neq 0$ then $F_0(A) / F_{-1}(A) \neq 0$ $\cancel{\emptyset}$

so $k = 0$ and

$A / F_{l-1}(A) \cong C$

and $F_0(A) = F_{\ell-1}(A) \Big/ \underset{\underset{0}{\shortparallel}}{F_{-1}(A)}$

$\cong F_{\ell-1}(A) = B$

$$\therefore\ \begin{array}{ccccccccc} 0 \to & F_k(A) & \to & A & \to & C & \to 0 \\ & \cong & & & & \cong & \\ & G(A)_k & & & & G(A)_\ell & \\ & \cong & & & & & \\ & B & & & & & \end{array}$$

exercise: Check 3)–4) (2) follows from 1)) ∎

If $F$ is a filtration of chain complexes $(C_*, \partial)$ s.t. $(F_s C_*, \partial)$ is a subcomplex of $(C_*, \partial)$ $\forall s$

then $F$ induces a filtration on the homology $H(C_*, \partial)$

$$F_s H(C_*, \partial) = \text{image}(H_*(F_s C_*, \partial) \to H_*(C_*, \partial))$$

if $F$ is finite, then easy to see

$$H(C_*, \partial) = \bigcup F_s H(C_*, \partial) \text{ and}$$

$$\bigcap F_s H(C_*, \partial) = 0$$

**Th$^{\underline{m}}$ 2:**

let $F$ be a finite filtration of a chain complex $(C_*, \partial)$ as above, then there is an $E^1$-spectral sequence with

1) $E^1_{s,t} = H_{s+t}\left(F_s C / F_{s-1} C\right)$

2) $d^1$ is the connecting homomorphism in the long exact sequence of the triple $(F_s C, F_{s-1} C, F_{s-2} C)$

3) $G(H(C_*, \partial))_{s,t} = E^\infty_{s,t}$

for 2 recall

$$0 \to F_{s-1}C / F_{s-2}C \to F_s C / F_{s-2}C \to F_s C / F_{s-1} C \to 0$$

induces a long exact sequence

$$H_k(F_s C / F_{s-2} C) \to H_k(F_s C / F_{s-1} C) \xrightarrow{d^1} H_{k-1}(F_{s-1} C / F_{s-2} C)$$

Remark: don't need $F$ finite but just spectral sequence is convergent to $E^\infty$ and $F$ bounded below

Idea of Proof:

Start with $E^0_{s,t} = G_s(C_{s+t}) = F_s C_{s+t} / F_{s-1} C_{s+t}$

$\partial$ induces $d^0 : E^0_{s,t} \to E^0_{s,t-1}$

set $E^1_{s,t} = H_{s+t}(G_s(C_*))$

given $\alpha \in E^1_{s,t}$, $\exists$ a chain $x \in F_s(C_{s+t})$

that represents $\alpha$

we know $d'\alpha = \partial \alpha = 0$ in $F_s C_{s+t-1} / F_{s-1} C_{s+t-1}$

in particular $\partial x \in F_{s-1} C_{s+t-1}$

define $d' : E'_{s,t} \to E'_{s-1,t} : \alpha = [x] \mapsto [\partial x]$

exercise: $d'$ is as claimed in theorem and $(d')^2 = 0$

now set $$E^2_{s,t} = \frac{\ker(d' : E'_{s,t} \to E'_{s-1,t})}{\operatorname{im}(d' : E'_{s+1,t} \to E'_{s,t})}$$

exercise: 1) show we could alternately define

$$E^2_{s,t} = \frac{\{c \in F_s C_{s+t} \mid \partial c \in F_{s-2}(C_{s+t-1})\}}{F_{s-1} C_{s+t} + \partial(F_{s+1} C_{s+t+1})}$$

intersect with numerator

2) show $\partial$ maps $E^2_{s,t}$ to $E^2_{s-2,t+1}$

in general define

$$E^r_{s,t} = \frac{\{c \in F_s(C_{s+t}) \mid \partial c \in F_{s-r}(C_{s+t-1})\}}{F_{s-1} C_{s+t} + \partial(F_{s+r-1} C_{s+t+1})}$$

really $\cap$ with numerator

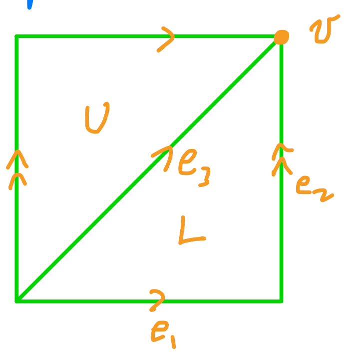
one can show $\partial$ induces a map

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

and $E^{r+1}_{s,t}$ is the homology of $(E^r_{s,t}, d^r)$

and $E^{\infty}_{s,t} = G(H(C_*, \partial))_{s,t}$

example 0:




$X_0 = \{v\}$
$X_1 = X_0 \cup e_1$
$X_2 = X_1 \cup e_2$
$X_3 = X_2 \cup e_3$
$X_4 = X_3 \cup U$
$X_5 = X_4 \cup L$

so $C^{CW}_*(T^2)$ has fitration

$$F_s C_* = C_*(X_s)$$

and

$$E^1_{s,t} = H_{s+t}\left(\frac{C_* X_s}{C_* X_{s-1}}\right)$$

$$= H_{s+t}(X_s, X_{s-1})$$

def$^n$ relative homology

$E^1_{s,t}$

| | | | | | | |
|---|---|---|---|---|---|---|
| $t=0$ | $H_0(X_0, X_1)$ | $H_1(X_1, X_0)$ | $H_2(X_2, X_1)$ | | | |
| $t=-1$ | | $H_0(X_1, X_0)$ | $H_1(X_2, X_1)$ | $H_2(X_3, X_2)$ | | |
| $t=-2$ | | | $H_0(X_2, X_1)$ | $H_1(X_3, X_2)$ | $H_2(X_4, X_3)$ | |
| $t=-3$ | | | | $H_0(X_3, X_2)$ | $H_1(X_4, X_3)$ | $H_2(X_5, X_4)$ |

$d^1 = 0$

$d^1 U = e_3$

| | | | | | |
|---|---|---|---|---|---|
| $\langle v \rangle \xleftarrow{d^1=0}$ | $\langle e_1 \rangle$ | $0$ | | | |
| | $0$ | $\langle e_2 \rangle$ | $0$ | | |
| | | $0$ | $\langle e_3 \rangle \leftarrow$ | $\langle U \rangle$ | |
| | | | $0$ | $0$ | $\langle L \rangle$ |

since $\tilde{H}_0(X_1/X_0)$

$E^2_{s,t}$

$$\begin{array}{cccccc}
\langle v\rangle & \langle e_1\rangle & 0 & & & \\
 & & \langle e_2\rangle & 0 & & \\
 & & 0 & 0 & 0 & \\
 & & & & 0 & \langle L\rangle
\end{array}$$

$d^2=0$

$E^3_{s,t}$

$$\begin{array}{cccccc}
\langle v\rangle & \langle e_1\rangle & 0 & & & \\
 & 0 & \langle e_2\rangle & 0 & & \\
 & & 0 & 0 & 0 & \\
 & & & & 0 & \langle L\rangle
\end{array}$$

really $E^3_{2,-1} = \dfrac{\{c\in F_2 : \partial c \in F_{-1}\}}{F_1C_1 + \underbrace{\partial F_4 C_2}}$

$d^3=0$

$\partial U = e_3 - e_1 - e_2$

$\partial L = e_1 + e_2 - e_3$

so $d^3 L = 0$

$E^4 = E^\infty$

$$\begin{array}{cccccc}
\langle v\rangle & \langle e_1\rangle & 0 & & & \\
 & 0 & \langle e_2\rangle & 0 & & \\
 & & 0 & 0 & 0 & \\
 & & & & 0 & \langle L\rangle
\end{array}$$

really $E^4_{1,0} = \dfrac{\{c\in F_1C_1 : \partial c \in F_{-3}\}}{F_0C_1 + \partial F_4 C_2}$

so again $d^4=0$

$d^4=0$

so by lemma 1 $H_0(T^2) = \langle v\rangle \cong \mathbb{Z}$

$H_2(T^2) = \langle L\rangle \cong \mathbb{Z}$

$$0 \to \mathbb{Z} \to H_1(T^2) \to \mathbb{Z} \to 0$$

so $H_1(T^2) = \mathbb{Z}\oplus\mathbb{Z}$

## example 1:

let $A \subset X$ be a subspace

the we get a filtration of $C_*(X)$ ← singular chain complex

$$F_1 C = C_*(X)$$
$$F_0 C = C_*(A)$$
$$F_{-1} C = 0$$

so we get a spectral sequence with

$$E^1_{s,t} = H_{s+t}\left( {F_s C_*}/{F_{s-1} C_*} \right)$$
$$\cong \begin{cases} H_{s+t}(X,A) & s=1 \\ H_{s+t}(A) & s=0 \\ 0 & s \neq 0,1 \end{cases}$$

and $E^1$ is

$$\begin{array}{ccccc}
\cdots & 0 & H_2(A) \xleftarrow{d^1} & H_3(X,A) & 0 \cdots \\
\cdots & 0 & H_1(A) \xleftarrow{d^1} & H_2(X,A) & 0 \cdots \\
\cdots & 0 & H_0(A) \xleftarrow{d^1} & H_1(X,A) & 0 \cdots \\
 & & 0 \xleftarrow{0} & H_0(X,A) & 0 \cdots \\
 & & 0 & 0 & 0 \\
 & & \vdots & \vdots & \vdots
\end{array}$$

$d^1 = \partial$ in L.E.S. of pair, or triple

$E^2$ (and $E^k$ $\forall k > 1$) is

$$\begin{array}{cc}
\vdots & \vdots \\
H_1(A)/\mathrm{im}\,\partial & \ker \partial \\
H_0(A)/\mathrm{im}\,\partial & \ker \partial \\
0 & H_0(X,A)
\end{array}$$

$s+t = \text{const}$

now $G(H(C_*(X)))_k = \bigoplus_{s+t=k} E^\infty_{s,t}$

$$= E^\infty_{0,k} \oplus E^\infty_{1,k-1}$$

$$= H_k(A) / \mathrm{im}(\partial : H_{k+1}(X,A) \to H_k(A))$$

$$\oplus \ker(\partial : H_k(X,A) \to H_{k-1}(A))$$

so lemma 1 gives

$$0 \to H_k(A)/\mathrm{im}\,\partial \to H_k(X) \to \ker \partial \to 0$$

which is same as

$$0 \to \mathrm{im}(\partial : H_{k+1}(X,A) \to H_k(A)) \to H_k(A) \to H_k(X)$$

$$\to \ker(\partial : H_k(X,A) \to H_{k-1}(A)) \to 0$$

so $\mathrm{im}\,\partial = \ker \imath_*$ $\quad \imath : A \to X$ inclusion

$\ker \partial = \mathrm{im}\, \jmath_*$ $\quad \jmath : X \to (X,A)$

this is the "hard part" of the L.E.S. of a pair so spectral sequences generalize this!

example 2: "simple" proof singular = cellular homology

If $X$ is a CW-complex, then we can filter the singular chain complex of $X$ using skeleta

$$F_k C = C_*(X^{(k)})$$

Thm 2 says there is a spectral sequence that converges to

$$E^{\infty}_{s,t} = G(H(C_*, \partial))_{s,t}$$

with

$$E^1_{s,t} = H_{s+t}\left( {}^{F_s C}/_{F_{s-1} C} \right)$$

$$= H_{s+t}\left( {}^{C_*(X^{(s)})}/_{C_*(X^{(s-1)})} \right)$$

$$= H_{s+t}( C_*( X^{(s)}, X^{(s-1)}))$$

$$= H_{s+t}(X^{(s)}, X^{(s-1)}) = H_{s+t}( {}^{X^{(s)}}/_{X^{(s-1)}})$$

$$= H_{s+t}(\text{wedge of } s\text{-spheres})$$

$$= \begin{cases} \bigoplus_{s\text{-cells}} \mathbb{Z} & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$= \begin{cases} C_s^{cw}(X) & t = 0 \\ 0 & t \neq 0 \end{cases}$$

cellular chain group

and sequence has $d^1 = \partial$ map in L.E.S of $(F_s, F_{s-1}, F_{s-2})$

i.e. $H_s(X^{(s)}, X^{(s-1)}) \xrightarrow{d^1} H_s(X^{(s-1)}, X^{(s-2)})$

in alg top class one shows $d^1 = \partial^{cw}$

so we have

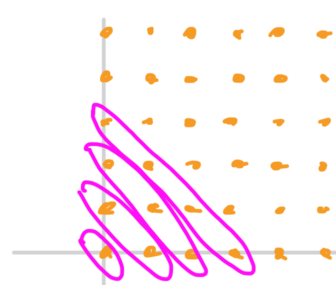
$$E^1 \quad \begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & & \\ & 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & & \\ \cdots & 0 & C_0^{cw}(X) & \xleftarrow{d^1} C_1^{cw}(X) & \xleftarrow{d^1} C_2^{cw}(X) & \xleftarrow{d^1} & \cdots \\ & 0 & 0 & 0 & 0 & & \\ & \vdots & \vdots & \vdots & \vdots & & \end{array}$$

then $E^2$ is

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & \cdots & \\ 0 & H_0^{cw}(X) & H_1^{cw}(X) & H_2^{cw}(X) & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \end{array}$$

so $E^\infty_{s,t} = \begin{cases} H_s^{cw}(X) & t = 0 \\ 0 & t \neq 0 \end{cases}$

and $G(H_*(X))_p = \bigoplus_{s+t=p} E^\infty_{s,t}$



lemma 1 part 5 $\Rightarrow$ $H_p(X) = H_p^{cw}(X)$

all of the above works for cohomology as well
and the differential respects the product structure

an $E_k$-cohomology spectral sequence $E$ is a sequence $\{E_r, d_r\}$ for $r \geq k$ such that

1) $E_r$ is a bigraded module and $d_r$ is a differential of bidegree $(r, -r+1)$

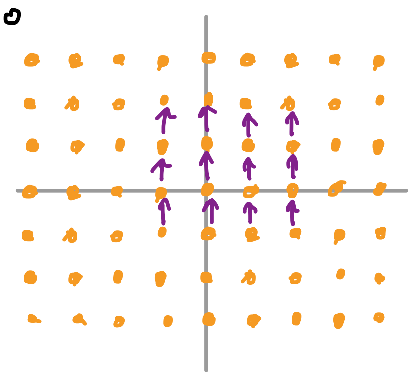
2) $d_r$ is a derivation of the algebra

$$d_r(a \cup b) = (d_r a) \cup b + (-1)^{|a|} a \cup (d_r b)$$

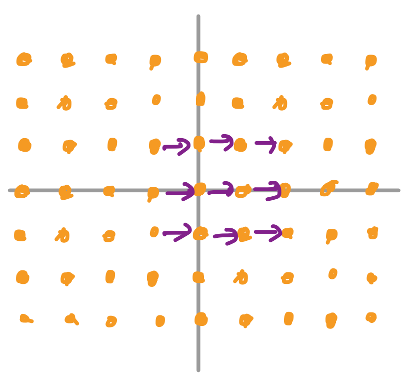
3) $E_{r+1} = H^*(E_r, d_r)$

example:

$d^0$



$d^1$



$d^2$